

Dual multiplets in $N=4$ superconformal mechanics

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Abstract

We propose Lagrangian formulations of a three-particles translation invariant $N = 4$ superconformal mechanics based on the standard and twisted $(1, 4, 3)$ supermultiplets. We show that in the appropriate set of coordinates, in which the bosonic kinetic terms for each of the two cases take conformally-flat forms, the corresponding models produce just similar physical systems with the flipped angular part of bosonic potential $U(x) \rightarrow 1/U(x)$. This flipping looks so simple only being written in the “angle” variable, while in the standard variables it looks more complicated to preserve the superconformal symmetry. We demonstrate at both the superfield and the component level, how familiar $N = 4$ supersymmetric 3-particles models, including 3 – *Calogero*, BC_2 and G_2 ones, can be constructed with twisted supermultiplets. We also present some new explicit examples of 3-dimensional superconformal mechanics.

1 Introduction

The recent interest in the supersymmetric extension of conformal mechanics [1] is motivated by its connection with the four-dimensional theory including $N = 2$ SUGRA in the context of black hole physics [2, 3, 4], as well as with the superextension of some one-dimensional integrable systems [5, 6, 7, 8, 9].

In the case of one-dimensional $N = 4$ supersymmetry there is a rich set of supermultiplets each containing four fermionic degrees of freedom but a different number of physical bosons (for a review see e.g. [10]). They admit $N = 4$ superconformal invariance under the general $D(2, 1; \alpha)$ superconformal group [11] in the standard superspace, as it was shown in many papers [12, 13, 14].

A particular class of $N = 4$ superconformal mechanics, which arises in superextensions of integrable models such as the Calogero one, has been intensively investigated recently [15, 16, 17, 18, 19, 20]. Studying these models has led to a connection between them and a system of non-linear differential WDVV equations [21, 22, 23, 24, 25, 26].

Most of these superconformal models were based on the standard $N = 4$ supermultiplet [27] whose physical boson was recognized as a dilaton field. This supermultiplet is transformed as a vector, under the action of the $SU(1, 1|2)$ superconformal group, which is in agreement with the rule of conservation of defining constraints under superconformal transformations. On the other hand, there exists another twisted (or "mirror") supermultiplet [28] with the same component field content as the standard one, but with different properties under superconformal transformations of the same group. This supermultiplet was also described in [29] in the framework of Harmonic Superspace (HSS) [30], which was formulated for the one-dimensional case in [31, 32].

In our previous consideration of the three-particles $N = 4$ supersymmetric *Calogero* model [17] we used the standard $N = 4$ supermultiplet and constructed within it the superconformal extension of some integrable models in superspace, for a particular choice of the parameter $\alpha = -1$ that corresponds to the $SU(1, 1|2)$ superconformal group. The main subject of the present paper is to demonstrate the equivalence of three-particles models with translational isometry (or two-particles ones without the latter) whose superfield actions are based either on standard [27] or twisted [28] $(1, 4, 3)$ supermultiplets. The existence of such an equivalence is not surprising, because the translation invariance, together with the (super)conformal one, fixed the 3-particle action, up to one arbitrary function which defined the model completely. Then one may properly choose this arbitrary function to reproduce some desired system. The most interesting feature of the approach with different supermultiplets is that starting with the (almost) same superspace action and performing just the replacement *standard* \leftrightarrow *twisted* superfield one can obtain similar systems with flipped angular part of bosonic potential $U(x) \leftrightarrow 1/U(x)$. Thus, these interchanging of the supermultiplets is indeed related with the duality transformation.

We start with the description of the realization of the $SU(1, 1|2)$ superconformal group in $N = 4$ superspace [14] and find how the standard and twisted supermultiplets are transformed under the action of this group demanding the preservation of the defining constraints. In Section 3 we construct $SU(1, 1|2)$ superconformal invariant actions for both of these supermultiplets in superspace. Then, passing to the on-shell form of each of the actions, we demonstrate that, in a proper set of new bosonic and fermionic variables, the corresponding actions lead to similar three-particles models with the flipped angular part of bosonic potential $U(x) \rightarrow 1/U(x)$. We also show how to choose the metric functions (super potentials), as well as the coupling constants, to reproduce the same systems within the approach with standard and twisted supermultiplets. Finally, we give some explicit examples of (super)conformal invariant models, including the 3 – *Calogero*, BC_2 and G_2 ones, as well as some new models.

2 $SU(1, 1|2)$ superconformal invariant action

In the present paper we consider the $N = 4$ superconformal mechanics based on two supermultiplets V and Φ , which are known as standard [27] and twisted [28] ones, respectively. Each of them has the same component content $(1, 4, 3)$ and their general supersymmetric actions lead to similar models at the component level. We will call these multiplets dual, because the resulting on-shell actions are related via a duality transformation of the bosonic potential $U \leftrightarrow 1/U$. On the other hand, the superconformal properties of the multiplets are different. In this section we show that V transforms as a vector under the $SU(1, 1|2)$ supergroup, while Φ is a scalar under the action of the latter.

2.1 $SU(1, 1|2)$ superconformal group in superspace and super-dilaton

We consider the real $N = 4$ superspace $\mathbf{R}^{(1|4)} = (t, \theta_a, \bar{\theta}^a)$, in which the spinor derivatives satisfy the standard anticommutation relations

$$\{D^a, \bar{D}_b\} = 2i\delta_b^a \partial_t, \quad \{D^a, D^b\} = \{\bar{D}_a, \bar{D}_b\} = 0. \quad (2.1)$$

In this superspace one may realize the most general $D(2, 1; \alpha)$ $N = 4$ superconformal group [11]. In what follows, we will deal only with the special case of $D(2, 1; \alpha)$ with $\alpha = -1$, which corresponds to the $SU(1, 1|2)$ superconformal group. This supergroup acts on the coordinates of $\mathbf{R}^{(1|4)}$ as [14, 27]

$$\delta t = E - \frac{1}{2}\theta_a D^a E - \frac{1}{2}\bar{\theta}^a \bar{D}_a E, \quad \delta \theta_a = -\frac{i}{2}\bar{D}_a E, \quad \delta \bar{\theta}^a = -\frac{i}{2}D^a E, \quad (2.2)$$

where the introduced superfunction $E(t, \theta, \bar{\theta})$ collects all parameters of $SU(1, 1|2)$ transformations

$$E = f(t) - 2i(\varepsilon_a \bar{\theta}^a - \theta_a \bar{\varepsilon}^a) + \theta^a \bar{\theta}^b B_{(ab)} + 2(\dot{\varepsilon}_a \bar{\theta}^a + \theta_a \dot{\bar{\varepsilon}}^a)(\theta \bar{\theta}) + \frac{1}{2}(\theta \bar{\theta})^2 \ddot{f}. \quad (2.3)$$

Here, the bosonic $f(t)$ and fermionic $\varepsilon^a(t)$ functions are restricted to be

$$f = a + bt + ct^2, \quad \varepsilon^a = \epsilon^a + t\zeta^a. \quad (2.4)$$

In (2.3), (2.4) the bosonic parameters a, b, c and $B_{(ab)}$ correspond to translations, dilatations, conformal boosts and rigid $SU(2)$ rotations, while the fermionic ones ϵ^a and ζ^a are the parameters of Poincaré and conformal supersymmetries, respectively.

It is rather important that the spinor derivatives transform through themselves under the $SU(1, 1|2)$ defined in (2.2), as [14, 27]

$$\delta D^a = \frac{i}{2}(D^a \bar{D}_b E) D^b, \quad \delta \bar{D}_a = \frac{i}{2}(\bar{D}_a D^b E) \bar{D}_b. \quad (2.5)$$

In order to construct a superconformal invariant action, one has to take into account that the superspace integration measure, defined as

$$ds = dt d^2\theta d^2\bar{\theta}, \quad (2.6)$$

transforms, in accordance with (2.2), under the $SU(1, 1|2)$ as

$$\delta ds = -\partial_t E ds. \quad (2.7)$$

Therefore, in order to maintain $SU(1, 1|2)$ invariance, one has to introduce the super-dilaton – $N = 4$, $d = 1$ superfield Y which transforms as follows¹

$$\delta Y = \partial_t E \cdot Y. \quad (2.8)$$

Now, one defines the superconformal invariant measure in the following way

$$\triangle s = Y ds. \quad (2.9)$$

The super-dilaton Y has to be further constrained by the irreducibility conditions

$$[D^a, \bar{D}_a] Y = 0. \quad (2.10)$$

These constraints are fully compatible with the transformation properties (2.5), (2.8).

As follows from (2.10), the super-dilaton Y contains the set of component fields, that is $(1, 4, 3)$, defined by

$$y = Y|, \quad \lambda^a = iD^a Y, \quad \bar{\lambda}_a = i\bar{D}_a Y, \quad Y_{(ab)} = D_{(a} \bar{D}_{b)} Y. \quad (2.11)$$

Besides them, there is also an additional constant parameter M arising as a corollary of the constraints (2.10)

$$\frac{\partial}{\partial t} D^2 Y = 0, \quad \frac{\partial}{\partial t} \bar{D}^2 Y = 0 \quad \Rightarrow \quad D^2 Y = iM, \quad \bar{D}^2 Y = -iM. \quad (2.12)$$

¹Sometimes, the name super-dilaton is used for the superfield $X = \text{Log} Y$.

Since this real constant is not of our interest at present, we will take it equal to zero.

The invariant action for the super-dilaton Y is unambiguously restored to be

$$S_Y = - \int dt d^2\theta d^2\bar{\theta} Y \log Y. \quad (2.13)$$

Having the invariant measure (2.9), the superconformal invariant action for every matter superfield, say \mathcal{A} , can be constructed as [17]

$$S = - \int \Delta s G(\mathcal{A}), \quad (2.14)$$

provided the superfield \mathcal{A} is a scalar under superconformal transformations.

To describe a three particle system with translation invariance one has to introduce three superfields: Y_0 for the center of mass coordinate, the super-dilaton Y , and one matter superfield \mathcal{A} . Clearly, the translation invariance results in the decoupling of Y_0 from the remaining coordinates. Therefore, to provide the super-conformal invariance the superfield Y_0 has to be a super-dilaton one obeying (2.8), (2.10). The second super-dilaton field Y is needed to construct a superconformal invariant action for the matter field (2.14). Thus, the most freedom we have is the choice of the matter superfield \mathcal{A} with the only property to be a scalar under (2.2). Summarizing, we conclude that the most general three-particles action possessing translation and superconformal invariance reads

$$S = - \int dt d^2\theta d^2\bar{\theta} (Y_0 \log Y_0 + Y \log Y + Y G(\mathcal{A})). \quad (2.15)$$

In what follows, we will deal mostly with the two last terms in (2.15) due to the obvious triviality of the action for the center of mass.

2.2 The standard and twisted (1, 4, 3) supermultiplets of $N = 4$ mechanics

As explained above, the direct way of constructing the invariant superconformal action is to use, together with the super-dilaton Y , some superfield \mathcal{A} which is a scalar under the $SU(1, 1|2)$ supergroup. The $N = 4$ standard [27] and twisted [28] supermultiplets with the component contents (1, 4, 3), denoted, respectively, as V and Φ can be described by the following constrained superfields in $N = 4$ superspace $\mathbf{R}^{(1|4)}$ ²

$$[D^a, \bar{D}_a]V = 0, \quad (2.16)$$

$$D^{(a} \bar{D}^{b)} \Phi = 0. \quad (2.17)$$

One can check that the constraints (2.16) and (2.17) are invariant under the $N = 4$ superconformal group $SU(1, 1|2)$ if the superfields V and Φ transform as:

$$\delta V = \partial_t E \cdot V, \quad \delta \Phi = 0. \quad (2.18)$$

From (2.18) one concludes that the superfield V , similarly to the super-dilaton Y one, has a vector-type transformation under the action of the $SU(1, 1|2)$ supergroup, while Φ has a scalar one. Thus, one may directly use the superfield Φ as the matter superfield with the action (2.14), while in the case of V one has to introduce the following scalar quantity including the super-dilaton Y [17]

$$Z = \frac{V}{Y} \Rightarrow S = - \int dt d^2\theta d^2\bar{\theta} Y G(Z). \quad (2.19)$$

As follows from (2.16) and (2.17), each of these superfields contains one physical boson, four fermions and three bosonic auxiliary components. A convenient definition of the components can be taken as

$$\begin{aligned} z &= Z|, \quad \phi = \Phi|, \\ \psi^a &= iD^a Z, \quad \bar{\psi}_a = i\bar{D}_a Z, \quad \xi^a = iD^a \Phi, \quad \bar{\xi}_a = i\bar{D}_a \Phi, \\ A_{(ab)} &= D_{(a} \bar{D}_{b)} V, \quad B = D^a D_a \Phi, \quad \bar{B} = \bar{D}_a \bar{D}^a \Phi, \quad C = -\frac{i}{4} [D^a, \bar{D}_a] \Phi. \end{aligned} \quad (2.20)$$

²The description of these supermultiplets in the framework of the harmonic superspace approach was given in [29, 32].

There is an additional constant parameter, which may be present, in virtue of constraints (2.16), in V

$$D^2V = im_v, \quad \bar{D}^2V = -im_v. \quad (2.21)$$

In what follows, this parameter should be identified with the coupling constant of the models considered below. Note that, in contrast to the superfield V , the multiplet Φ does not contain a similar constant quantity. In that case it arises after adding to the general action the Fayet-Iliopoulos terms (FI-terms) (next Section), or performing the dualization procedure to one of the auxiliary components entering Φ .

3 On the equivalence of the component actions

In this section we show that the invariant $SU(1,1|2)$ superconformal models based on standard and twisted supermultiplets describe the same physical systems. After obtaining the corresponding component actions, one finds how one of them goes into another, by the proper change of variables and vice versa.

Let us start with the superconformal action of the standard multiplet which is represented by the superfield Z (2.19) [17]

$$S_{(Y,Z)} = -\frac{1}{2} \int ds \left\{ Y \log Y + Y F(Z) \right\}. \quad (3.1)$$

After integrating over θ 's and using (2.20), one gets the off-shell component action ³

$$\begin{aligned} S_{(Y,Z)} = & \frac{1}{32} \int dt \left\{ \frac{4}{y} \dot{y}^2 + \frac{4i}{y} (\dot{\lambda}^a \bar{\lambda}_a - \lambda^a \dot{\bar{\lambda}}_a) - \frac{2}{y} Y^2 - \frac{4}{y^2} \lambda_a Y^{(ab)} \bar{\lambda}_b - \frac{2}{y^3} \lambda^2 \bar{\lambda}^2 \right. \\ & + 4y F'' z^2 + 4iy F'' (\dot{\psi}^a \bar{\psi}_a - \psi^a \dot{\bar{\psi}}_a) - 4i F'' \dot{z} (\psi^a \bar{\lambda}_a - \lambda^a \bar{\psi}_a) \\ & - \frac{1}{y} F'' m_v^2 + \frac{2i}{y} F'' (\psi^a \lambda_a - \bar{\lambda}_a \bar{\psi}^a) m_v - i F''' (\psi^2 - \bar{\psi}^2) m_v \\ & - \frac{2}{y} F'' (A^{(ab)} - z Y^{(ab)}) (A_{(ab)} - z Y_{(ab)}) - 4F'' \psi_a Y^{(ab)} \bar{\psi}_b \\ & - \frac{4}{y} F'' \left[\psi_a (A^{(ab)} - z Y^{(ab)}) \bar{\lambda}_b + \lambda_a (A^{(ab)} - z Y^{(ab)}) \bar{\psi}_b \right] + 4F''' \psi_a (A^{(ab)} - z Y^{(ab)}) \bar{\psi}_b \\ & \left. - \frac{2}{y} F'' (\lambda^2 \bar{\psi}^2 + \psi^2 \bar{\lambda}^2) - \frac{8}{y} F'' \lambda^a \psi_a \bar{\lambda}_b \bar{\psi}^b + 4F''' (\lambda^a \psi_a \bar{\psi}^2 + \psi^2 \bar{\lambda}_a \bar{\psi}^a) - y F^{(4)} \psi^2 \bar{\psi}^2 \right\}, \end{aligned}$$

where

$$F'' = \left. \frac{\partial^2 F}{\partial Z \partial \bar{Z}} \right|_{\theta=0}, \quad (3.2)$$

and the following rules of summation are used: $\lambda^2 = \lambda^a \lambda_a$, $\bar{\lambda}^2 = \bar{\lambda}_a \bar{\lambda}^a$, etc. Eliminating the auxiliary bosonic fields $A_{(ab)}$, $Y_{(ab)}$ and making the following change of variables

$$\begin{aligned} F'' = 4 \left(\frac{dx}{dz} \right)^2 &= h^2(x), \quad u = \frac{1}{2} \log y, \\ \lambda^a = 2e^u \eta^a, \quad \bar{\lambda}_a = 2e^u \bar{\eta}_a, \quad \psi^a &= \frac{2e^{-u}}{h} \gamma^a, \quad \bar{\psi}_a = \frac{2e^{-u}}{h} \bar{\gamma}_a, \end{aligned} \quad (3.3)$$

one obtains the on-shell Lagrangian in which the kinetic bosonic part is of the conformally-flat form, while the kinetic terms for fermions has the standard structure of a free action

$$\begin{aligned} \mathcal{L}_{stand} = & \frac{1}{2} e^{2u} (\dot{u}^2 + \dot{x}^2) + \frac{i}{2} (\dot{\eta}^a \bar{\eta}_a - \eta^a \dot{\bar{\eta}}_a) + \frac{i}{2} (\dot{\gamma}^a \bar{\gamma}_a - \gamma^a \dot{\bar{\gamma}}_a) - i \dot{x} (\gamma^a \bar{\eta}_a - \eta^a \bar{\gamma}_a) \\ & - \frac{1}{32} e^{-2u} h^2 m_v^2 + \frac{i}{4} e^{-2u} h (\gamma^a \eta_a - \bar{\gamma}_a \bar{\eta}^a) m_v - \frac{i}{8} e^{-2u} h_x (\gamma^a \gamma_a - \bar{\gamma}_a \bar{\gamma}^a) m_v \end{aligned} \quad (3.4)$$

³The superspace measure is defined to be: $d^2\theta d^2\bar{\theta} = \frac{1}{16} D^a D_a \bar{D}_a \bar{D}^a$

$$\begin{aligned}
& - \frac{e^{-2u}}{4} \left(\gamma^2 \bar{\eta}^2 + \eta^2 \bar{\gamma}^2 \right) + \frac{e^{-2u}}{2} \frac{h_x}{h} \left(\gamma^a \eta_a \bar{\gamma}^2 + \gamma^2 \bar{\gamma}_a \bar{\eta}^a \right) - e^{-2u} \eta^a \gamma_a \bar{\eta}_b \bar{\gamma}^b \\
& + \frac{e^{-2u}}{4} \left[3 + \frac{h_x^2}{h^2} - \frac{h_{xx}}{h} \right] \gamma^2 \bar{\gamma}^2 - \frac{e^{-2u}}{4} \eta^2 \bar{\eta}^2.
\end{aligned}$$

The superconformal action of the twisted multiplet Φ can be written as the following integral in $N = 4$ superspace

$$S_{(Y,\Phi)} = \frac{1}{2} \int ds \left\{ -Y \log Y + Y G(\Phi) \right\}. \quad (3.5)$$

It produces, after integrating over Grassmann variables with the help of (2.20), the off-shell component action

$$\begin{aligned}
S_{(Y,\Phi)} &= \frac{1}{32} \int dt \left\{ \frac{4}{y} \dot{y}^2 + \frac{4i}{y} (\dot{\lambda}^a \bar{\lambda}_a - \lambda^a \dot{\bar{\lambda}}_a) - \frac{2}{y} Y^2 - \frac{4}{y^2} \lambda_a Y^{(ab)} \bar{\lambda}_b - \frac{2}{y^3} \lambda^2 \bar{\lambda}^2 \right. \\
&+ 4y G'' \dot{\phi}^2 + 4iy G'' (\dot{\xi}^a \bar{\xi}_a - \xi^a \dot{\bar{\xi}}_a) - 4i G'' \dot{\phi} (\xi^a \bar{\lambda}_a - \lambda^a \bar{\xi}_a) \\
&+ y G'' (B \bar{B} - 4C^2) - 2G'' (\xi^a \lambda_a \bar{B} + \bar{\lambda}_a \bar{\xi}^a B) - 4i G'' (\xi^a \bar{\lambda}_a + \lambda^a \bar{\xi}_a) C \\
&- 4G'' \xi_a Y^{(ab)} \bar{\xi}_b - y G''' \xi^2 \bar{B} - y G''' \bar{\xi}^2 B - 4iy G''' \xi^a \bar{\xi}_a C \\
&+ 2G''' \xi^2 \bar{\xi}^a \bar{\lambda}^a + 2G''' \xi^a \lambda_a \bar{\xi}^2 + y G^{(4)} \xi^2 \bar{\xi}^2 \left. \right\}, \quad (3.6)
\end{aligned}$$

where

$$G'' = \left. \frac{\partial^2 G}{\partial \Phi \partial \Phi} \right|_{\theta=0}. \quad (3.8)$$

In contrast with the previous case we have no coupling constant in the action (3.6), because the invariance of the basic constraints for the superfield Φ (2.17) under superconformal symmetry forbids the appearance of any constant among its components. Therefore, the unique way to have the coupling constant in the action is to introduce an additional FI-term. Thus, in the case of the twisted supermultiplet our basic action (2.15) has to be modified as

$$S \rightarrow S + S_{(Y,\Phi)}^{FI}. \quad (3.9)$$

In general, one should add a different kind of FI-terms to the action $S_{(Y,\Phi)}$, each of them associated with all possible auxiliary fields present in it. But in order to preserve the superconformal invariance of the modified action (3.9) one has to choose the FI-terms in the form in which only the special combination of complex conjugated auxiliary fields is present (here m_ϕ is an arbitrary real constant):

$$S_{(Y,\Phi)}^{FI} = \frac{i}{32} \int dt m_\phi (B - \bar{B}). \quad (3.10)$$

Indeed, one may easily check that the integrand in (3.10) transforms as a full time derivative under the $SU(1,1|2)$ superconformal group. Now, excluding the auxiliary fields, in accordance with their equations of motions, in the action (3.9) and passing to following set of variables

$$\begin{aligned}
G'' &= 4 \left(\frac{dx}{d\phi} \right)^2 = g^2(x), \quad u = \frac{1}{2} \log y, \\
\lambda^a &= 2e^u \eta^a, \quad \bar{\lambda}_a = 2e^u \bar{\eta}_a, \quad \xi^a = \frac{2e^{-u}}{g} \rho^a, \quad \bar{\xi}_a = \frac{2e^{-u}}{g} \bar{\rho}_a.
\end{aligned} \quad (3.11)$$

one finally obtains the on-shell Lagrangian with the coupling constant m_ϕ :

$$\begin{aligned}
\mathcal{L}_{tw} &= \frac{1}{2} e^{2u} (\dot{u}^2 + \dot{x}^2) + \frac{i}{2} (\dot{\eta}^a \bar{\eta}_a - \eta^a \dot{\bar{\eta}}_a) + \frac{i}{2} (\dot{\rho}^a \bar{\rho}_a - \rho^a \dot{\bar{\rho}}_a) - i\dot{x} (\rho^a \bar{\eta}_a - \eta^a \bar{\rho}_a) \\
&- e^{-2u} \frac{m_\phi^2}{32g^2} + ie^{-2u} \frac{m_\phi}{4g} (\rho^a \eta_a - \bar{\rho}_a \bar{\eta}^a) + ie^{-2u} \frac{m_\phi}{8} \frac{g_x}{g^2} (\rho^a \rho_a - \bar{\rho}_a \bar{\rho}^a) \\
&- \frac{e^{-2u}}{4} (\rho^2 \bar{\eta}^2 + \eta^2 \bar{\rho}^2) - \frac{e^{-2u}}{2} \frac{g_x}{g} (\rho^a \eta_a \bar{\rho}^2 + \rho^2 \bar{\rho}_a \bar{\eta}^a) - e^{-2u} \eta^a \rho_a \bar{\eta}_b \bar{\rho}^b \\
&+ \frac{e^{-2u}}{4} \left[3 - \frac{g_x^2}{g^2} + \frac{g_{xx}}{g} \right] \rho^2 \bar{\rho}^2 - \frac{e^{-2u}}{4} \eta^2 \bar{\eta}^2.
\end{aligned} \quad (3.12)$$

The analysis of the explicit structure of the Lagrangians given in (3.4) and (3.12) leads to the statement that the $SU(1,1|2)$ superconformal models for standard and twisted multiplets are equivalent if the following relations between metric functions and coupling constants take place

$$h(x) = \frac{1}{g(x)}, \quad m_\phi = m_v. \quad (3.13)$$

Also, the fermionic component should be identified: $\gamma = \rho$.

Finally, let us note that one can also proceed to the Hamiltonian analysis of these systems constructing the corresponding momenta, supercharges, etc. The result of doing so, shows us that the Hamiltonians coincide, when the rules (3.13) are taken into account.

4 Examples of Superpotentials

4.1 Some examples of 3-particles models

As an equivalence of the superconformal models was obtained for the actions of standard and twisted $(1,4,3)$ multiplets, we have a reason to reproduce some known examples of three-particles systems such as the *Calogero* [5], BC_2 (with $g_1 = g_2 \equiv g$) [6, 7] and G_2 (with $f = g \equiv g$) [8] ones.

Let us remind [17] that the bosonic part of the actions for the following models - *Calogero* [5], BC_2 (with $g_1 = g_2 \equiv g_{BC}$) [6, 7] and G_2 (with $f = g \equiv g_G$) [8] models in the set of coordinates (u, x) can be written as

$$S_{\text{Cal}} = \int dt \left[\frac{1}{2} \sum_{i=1}^3 \dot{x}_i^2 - \sum_{i<j} \frac{2g}{(x_i - x_j)^2} \right] = \int dt \left[\frac{1}{2} \dot{X}_0^2 + \frac{1}{2} e^{2u} (\dot{u}^2 + \dot{x}^2) - e^{-2u} \frac{9g}{\cos^2 3x} \right], \quad (4.1)$$

$$S_{G_2} = \int dt \left[\frac{1}{2} \sum_{i=1}^3 \dot{x}_i^2 - \sum_{i<j} \frac{2g}{(x_i - x_j)^2} - \sum_{i<j; i,j \neq k} \frac{6g}{(x_i - x_j + 2x_k)^2} \right] = \int dt \left[\frac{1}{2} \dot{X}_0^2 + \frac{1}{2} e^{2u} (\dot{u}^2 + \dot{x}^2) - e^{-2u} \frac{36g}{\sin^2 6x} \right], \quad (4.2)$$

$$S_{BC_2} = \int dt \left[\frac{1}{2} \sum_{i=1}^2 \dot{y}_i^2 - g \left(\frac{1}{(y_1 - y_2)^2} + \frac{1}{(y_1 + y_2)^2} + \frac{1}{2y_1^2} + \frac{1}{2y_2^2} \right) \right] = \int dt \left[\frac{1}{2} e^{2u} (\dot{u}^2 + \dot{x}^2) - e^{-2u} \frac{8g}{\sin^2 4x} \right]. \quad (4.3)$$

Here, in the cases (4.1) and (4.2)

$$X_0 = \frac{1}{\sqrt{3}} (x_1 + x_2 + x_3), \quad y_1 = \frac{1}{\sqrt{6}} (2x_1 - x_2 - x_3), \quad y_2 = \frac{1}{\sqrt{2}} (x_2 - x_3), \quad (4.4)$$

and, finally,

$$y_1 = e^u \sin x, \quad y_2 = e^u \cos x. \quad (4.5)$$

As shown in [17], the supersymmetric version of these actions (with a decoupled center of mass in the cases of (4.3) and (4.2)) can be constructed with the pair of superfields (Y, Z) - the standard $(1,4,3)$ multiplets, when the corresponding superfunctions $F(Z)$ are taken as

$$\begin{aligned} F_{\text{Cal}} &= \frac{2}{9} \left[(1+Z) \log(1+Z) + (1-Z) \log(1-Z) \right], \quad g = \frac{m_v^2}{648} \\ F_{G_2} &= \frac{1}{18} \left[(1+4Z) \log(1+4Z) + (1-4Z) \log(1-4Z) \right], \quad g = \frac{m_v^2}{648} \\ F_{BC_2} &= \frac{1}{8} \left[(1+4Z) \log(1+4Z) + (1-4Z) \log(1-4Z) \right], \quad g = \frac{m_v^2}{64}. \end{aligned} \quad (4.6)$$

Since the corresponding Lagrangians depend on the superfield Z , which transforms as a scalar under the $SU(1,1|2)$ superconformal group, we can replace this superfield with Φ and change the overall sign, in agreement with (3.1), (3.5). Thus, with these particular choices of superpotentials $G(\Phi)$ in (3.5) we obtain the following result for the bosonic part of the actions, after passing to polar coordinates (u, x) (3.11) ⁴

$$S_{\text{Cal}} = \int dt \left\{ \frac{1}{2} e^{2u} (\dot{u}^2 + \dot{x}^2) - 9g e^{-2u} \cos^2 3x \right\}, \quad \phi = \sin 3x, \quad (4.7)$$

$$S_{G_2} = \int dt \left\{ \frac{1}{2} e^{2u} (\dot{u}^2 + \dot{x}^2) - 36g e^{-2u} \sin^2 6x \right\}, \quad \phi = \frac{1}{4} \cos 6x, \quad (4.8)$$

$$S_{\text{BC}_2} = \int dt \left\{ \frac{1}{2} e^{2u} (\dot{u}^2 + \dot{x}^2) - 8g e^{-2u} \sin^2 4x \right\}, \quad \phi = \frac{1}{4} \cos 4x. \quad (4.9)$$

In order to reconstruct the actions in the set of Euclidean coordinates $(x_i, i = 1, 2, 3)$, one has to perform the same change of variables as in (4.4), (4.5).

It is also easy to find the structure of the superpotentials $G(\Phi)$ which correspond to the original *Calogero*, G_2 and BC_2 models:

$$G_{\text{Cal}}(\Phi) = \frac{4}{9} \log \left(1 + e^{-\frac{9}{2}\Phi} \right), \quad e^{\frac{9}{2}\phi} = \tan^2 \left(\frac{3}{2}x + \frac{\pi}{4} \right), \quad (4.10)$$

$$G_{G_2}(\Phi) = \frac{1}{9} \log \left(1 + e^{-\frac{9}{2}\Phi} \right), \quad e^{\frac{9}{2}\phi} = \tan^2(3x), \quad (4.11)$$

$$G_{\text{BC}_2}(\Phi) = \frac{1}{4} \log \left(1 + e^{-2\Phi} \right), \quad e^{2\phi} = \tan^2(2x). \quad (4.12)$$

Summarizing, we conclude that the superpotential having the same dependence on standard or twisted supermultiplets leads to the conformal three-particles models, the potential terms of which are inverse to each other. However, one should stress that the inverse form of the potential appears only in the “angle” variable x . Indeed, for example for the 3 – *Calogero* model we will have

$$U_{Y,Z} = \frac{9}{2} e^{-2u} \frac{1}{\cos^2(3x)} \rightarrow U_{Y,\Phi} = \frac{9}{2} e^{-2u} \cos^2(3x). \quad (4.13)$$

The same “flipping” of the potential, being rewritten in the standard x_i coordinates (4.4), (4.5) will read

$$U_{Y,Z} = \frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 - x_3)^2} + \frac{1}{(x_2 - x_3)^2} \rightarrow U_{Y,\Phi} = \frac{729}{16} \frac{(x_1 - x_2)^2 (x_2 - x_3)^2 (x_3 - x_2)^2}{(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_1 x_3 - x_2 x_3)^4}. \quad (4.14)$$

Thus, systems which look quite different are related by a special duality transformations. So, one may expect that integrability on the one side (say, for the 3 – *Calogero* model) will lead to the integrability on other side, where the potential looks completely different.

4.2 Generating function of three-particles models

The construction considered above has been connected with a certain kind of superpotential $F(Z)$ or $G(\Phi)$. Let us analyze which three-particle models can be obtained, proceeding from a generating function of the following kind

$$F(Z) = \alpha \left[(1 + \beta Z) \log(1 + \beta Z) + (1 - \beta Z) \log(1 - \beta Z) \right] \quad (4.15)$$

for the models based on the standard multiplets. This superfunction contains parameters α and β which can take any arbitrary values. So, for example, considering the case of (4.15), we find for the second derivatives of $F(Z)$ taken at the point $\theta = 0$ the following expression

$$F''(z) = 2\alpha\beta^2 \frac{1}{1 - (\beta z)^2}. \quad (4.16)$$

⁴Here ϕ denotes the physical bosonic component of the superfield Φ .

Performing the following change of variables ⁵

$$\beta z = \cos(nx), \quad \rightarrow \quad \frac{dz}{dx} = -\frac{n}{\beta} \sin(nx), \quad n \in \mathbf{N}, \quad (4.17)$$

one finds for the bosonic part of the kinetic terms

$$\begin{aligned} L^{\text{kin}} &= \frac{1}{32} \left(\frac{4}{y} \dot{y}^2 + 4y F'' \dot{z}^2 \right) = \frac{1}{2} e^{2u} \dot{u}^2 + \frac{1}{8} e^{2u} \left[\frac{2\alpha\beta^2}{1 - (\beta z)^2} \right] \Big|_{\beta z = \cos(nx)} \left(\frac{dz}{dx} \right)^2 \dot{x}^2 \\ &= \frac{1}{2} e^{2u} \left(\dot{u}^2 + \left[\frac{\alpha n^2}{2} \right] \dot{x}^2 \right), \end{aligned} \quad (4.18)$$

where, as well as above, $u = \frac{1}{2} \log y$. Thus, the bosonic kinetic term does not depend on β and takes the conformally-flat form, if the following relation is satisfied

$$\alpha n^2 = 2. \quad (4.19)$$

On the other hand, the potential term depends on the parameter β , and has the following expression in new coordinates

$$L^{\text{Pot}} = -\frac{e^{-2u}}{32} m_v^2 F''(z) \Big|_{\beta z = \sin(nx)} = -\frac{e^{-2u}}{32} \frac{2\alpha\beta^2}{\cos^2(nx)} m_v^2. \quad (4.20)$$

From the last expression one concludes that the value of β determines only the relation between the parameter m_v^2 and the coupling constant g of the corresponding model. Thus, in principle, one may put it equal to **one**: $\beta = 1$.

Summarizing, we will get that the values of the parameters α, n and the corresponding three-particles models with translational isometry (or the two-particles ones without the latter) are interconnected

n	2	3	4	5	6	...	$\forall n \in \mathbf{N}$
α	1/2	2/9	1/8	2/25	1/18	...	$2/n^2$
models	-	Cal	BC ₂	-	G ₂	...	-

if $\sqrt{\frac{2}{\alpha}} = n, n \in \mathbf{N}$.

These examples lead us to conclude that, in order to achieve at the component level the action of three-particles conformal invariant models, whose kinetic part is just a free action and the potential one is an arbitrary function of one of the coordinates, one can start with the superfield Lagrangian, as given in (4.15), without any restrictions on the parameter α . Then, requiring that this parameter satisfies some additional conditions like those in (4.19), one obtains the component actions which describe some $N = 4$ superconformal translational invariant three-particles models (or two-particles ones without such an invariance). In particular, all previously discussed models, including A_2 Calogero, BC_2 and G_2 [17], are contained in the superpotentials (4.15) by the appropriate value of the parameter α .

5 Conclusion

In the present paper we analyzed $N = 4$ supersymmetric mechanical models based on the standard and twisted multiplets, each of which has $(1, 4, 3)$ component field content. In both cases we constructed the superfield actions invariant under a particular choice of the $D(2, 1; \alpha)$ superconformal group with $\alpha = -1$, which corresponds to the $SU(1, 1|2)$ one. We demonstrated that, at the component level, these actions correspond to the same three-particles superconformal models, when the proper relation between the metric functions is performed and the coupling constants are proportional to each other. We also obtained the explicit form of the superpotential of

⁵On equal footing, one can choose $\beta z = \sin(nx)$.

some translational invariant three-particles models (or two-particles one without such an invariance), including, in particular, the A_2 Calogero one, as a proper function of the twisted supermultiplet. Moreover, a new structure of superpotentials, both for the standard and twisted $(1, 4, 3)$ supermultiplets, which leads to a class of three-particles models, was found. In general, these superpotentials depend on one real parameter α , and after fixing its value one gets the correct expression for the superconformal extension of the corresponding models.

One of the most interesting features of the constructed models is the “flipping” of the angular part of bosonic potential. If we start from the system with the standard superfields Y, Z and then replace the second scalar superfield Z by a twisted one Φ , then the resulting system will have just a “flipped” angular part of bosonic potential $U(x) \rightarrow 1/U(x)$. But this flipping looks so simple only being written in the “angle” variable, while in the standard variables it looks more complicated (4.14). It is worth to stress that the main property of the started system – $N = 4$ superconformal symmetry – is preserved by this duality transformation. Just this superconformal symmetry on both sides makes it impossible to flip the potential in the standard coordinates x_i , because $U(x_i)$ and $1/U(x_i)$ have different dilaton weights. That is why the correcting factor (the power of the denominator) appears in the explicit form (4.14). It is an interesting question to check whether the dual potential in (4.14) is still integrable, as it happens for the 3 – Calogero model.

The present consideration is just the first step in a more ambitious task, i.e. to consider the system with n multiplets of one type and m twisted ones. We hope that the additional freedom, which is related with the presence of two types of supermultiplets in the same action, will help to override the old problem of the construction of four- (and higher) particles Calogero models with $N = 4$ superconformal symmetry [22]-[26].

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